This book presents a good introduction to the several topics which it treats. However, the level of presentation is rather mixed, since about one third of the material assumes a knowledge of functional analysis.

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10 [4, 6].—IVAR STAKGOLD, *Boundary Value Problems of Mathematical Physics, Vol.* I, The Macmillan Company, New York, 1967, viii + 340 pp., 24 cm. Price $12.95.

This is the first of two volumes intended for a graduate course in mathematical physics. Although the topics discussed are mathematical in nature, it is written in a clear and pleasant style by a man who knows how to talk to physicists and engineers and who enjoys doing so.

While the easier results are proved, more difficult theorems or those requiring lengthy proof are motivated heuristically, in such a way that the reader at least gets the feeling of how the proof goes. In such cases it is clearly stated that a proof is needed, and whether the proof is difficult or easy.

Chapter 1 deals with ordinary differential equations. In addition to the standard material on this subject, there is a beautiful discussion of one-dimensional distribution theory. Its purpose is to provide a firm foundation for the Dirac delta function, which is then used to define fundamental solutions and Green's functions.

Chapter 2 is an introduction to linear spaces, with particular emphasis on linear transformations in a Hilbert space.

These concepts are applied in Chapter 3 to the study of linear integral equations with symmetric kernels. This chapter includes some discussion of variational methods both for nonhomogeneous problems and for eigenvalue problems. In particular, eigenfunction expansions are discussed, and the Rayleigh-Ritz method and the eigenvalue inclusion theorem are presented.

Chapter 4 deals with singular self-adjoint boundary value problems for second-order ordinary differential operators. It includes a proof of Weyl's limit point-limit circle theorem, and a discussion of the general spectral representation.

The author has been able to concoct a large set of exercises which are nontrivial and educational, but still not too difficult for students taking the course for which the book is designed.

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11 [4, 5, 7].—A. W. BABISTER, *Transcendental Functions Satisfying Nonhomogeneous Linear Differential Equations*, The Macmillan Company, New York, 1967, xi + 414 pp., 24 cm. Price $14.95.

Consider

$$(1) \qquad L(D)y(x) = f(x)\,, \qquad L(D) = \sum_{k=0}^{n} p_k(x)D^{n-k}\,, \qquad D = d/dx\,.$$

The general theory for obtaining particular solutions of (1) is treated in a number of texts and references. In the case where detailed properties of $y(x)$ are desired for specific operators $L(D)$ and functions $f(x)$, very little data is to be found in the literature. This is particularly the situation where

$$(2) \qquad M(\delta)y(x) = xf(x)\,,$$

$$(3) \qquad \begin{aligned} M(\delta) &= \delta(\delta + \rho_1 - 1)(\delta + \rho_2 - 1)\cdots(\delta + \rho_q - 1) \\ &\quad -x(\delta + \alpha_1)(\delta + \alpha_2)\cdots(\delta + \alpha_q)\,, \qquad \delta = xD\,, \end{aligned}$$

$$(4) \qquad M(\delta)\,{}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\,\middle|\, x\right) = 0\,, \qquad {}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\,\middle|\, x\right) = {}_pF_q\left(\begin{matrix}\alpha_1, \alpha_2, \cdots, \alpha_p\\ \rho_1, \rho_2, \cdots, \rho_q\end{matrix}\,\middle|\, x\right),$$

and $f(x)$ has the form

$$(5) \qquad f(x) = tx^{\gamma}(1 - ax)^{\epsilon}e^{\lambda x}$$

for special values of a, γ, ϵ and λ. Here t is free of x, and the ${}_pF_q$ is the notation for the generalized hypergeometric function. Throughout the discussion we follow the notation of [1].

Many of the special functions of mathematical physics are characterized by the ${}_pF_q$ symbol. The history of such functions and of (4) is long and varied. This includes not only theoretical developments stemming from the pure side of mathematics, but also the analysis of diverse physical problems where (4) appears. Physically, when the order of $M(\delta)$ is 2, $f(x)$ can be interpreted as a forcing function, and characterization of the ensuing response due to the presence of $f(x)$ is very important. Thus, for this and numerous other problems, the volume under review should provide a useful compendium of results.

The bulk of the present volume is concerned with solutions of (2) for the cases $p = 0,\ q - 1,\ p = q = 1$ and $p = 2,\ q = 1$. For each of these situations the order of $M(\delta)$ is 2. The volume contains other material on which we shall comment in due course.

There are 11 chapters. Each concludes with a set of problems, and where appropriate, problems are taken from physical applications.

Chapter 1 takes up general methods for the solution of (1), including variation of parameters, Cauchy's method, solution in terms of integrals, series expansions, and Green's functions. When the coefficients $p_k(x)$ in (1) are constant, then a solution is often facilitated by use of the Laplace transform. This is the subject of Chapter 2.

Perhaps the best known and oldest example of (2)–(5) is the case of Lommel functions. Thus with $p = 0$, $q = 1$, $\rho_1 = \nu + 1$, and appropriate transformations, we have

$$(6) \qquad (z^2D^2 + zD + z^2 - \nu^2)h(z) = z^{\mu+1}$$

which is satisfied by $s_{\mu,\nu}(z)$ and $S_{\mu,\nu}(z)$, see [1, Vol. 2, Chapter 6] or [2]. Both of the above functions satisfy the same difference and difference-differential formulas. Many other properties of these functions are known, e.g., integral representations, integrals involving them, and asymptotic expansions. The delineation of such properties is given in Chapter 3. As the homogeneous solutions of (6) are the Bessel functions $J_\nu(z)$ and $Y_\nu(z)$, some properties of these functions needed in the analysis are also developed. Actually, this chapter begins with the analysis of Struve's function, which is the special case $\mu = \nu$, since $s_{\nu,\nu}(z) = 2^{\nu-1}\Gamma(\frac{1}{2})\Gamma(\frac{1}{2} + \nu)H_\nu(z)$. The chapter then concludes with an analysis of the Lommel functions. Thus there is some repetition. Now $s_{\mu,\nu}(z)$ is essentially a ${}_1F_2$. Hence many of its properties are a consequence of the general theory for the ${}_pF_q$. However, for the most part this general approach is not followed. In this connection, if in (5) $f(x) = \mu(\mu + \rho_1 - 1) \times (\mu + \rho_2 - 1) \cdots (\mu + \rho_q - 1)x^{\mu-1}$, it can be readily shown that

$$y(x) = x^{\mu}{}_{p+1}F_{q+1}\left(\begin{matrix}1, \mu + \alpha_p \\ \mu + 1, \mu + \rho_q\end{matrix}\middle| x\right) \tag{7}$$

so that numerous properties of $y(x)$ follow on appeal to the general theory for the ${}_pF_q$. The relation (7) is given in Chapter 8.

The kinds of properties known for Lommel functions serve as models for properties sought for solutions of other nonhomogeneous differential equations studied in the text. Further, as is to be expected, properties of $y(x)$ as defined by (2) are akin to those for the homogeneous solutions for this equation.

Solutions of (2) with $p = q = 1$, $\alpha_1 = a$, $x = z$, $t^{-1}f(z) = e^{z/2}$, $e^{z/2}z^{1-c}$, $z^{\sigma-1}$ and $e^{\rho z}z^{\sigma-1}$ for appropriate t, that is

$$(zD^2 + (c - z)D - a)y(z) = f(z)\ , \tag{8}$$

are taken up in Chapter 4. Here the homogeneous solutions are the confluent hypergeometric functions ${}_1F_1(a; c; z)$ and $\psi(a; c; z)$, and properties of these functions are also given. The author seems unaware that the special case $t^{-1}F(z) = z^{\sigma-1}$, $c = 2a$ has been previously studied, see [3].

In (8), put $y(z) = z^{-c/2}e^{z/2}g(z)$, $c = 2u + 1$, $a = \frac{1}{2} - k + u$, then

$$[D^2 + (-\tfrac{1}{4} + k/z + (\tfrac{1}{4} - u^2)z^2)]\, g(z) = e^{-z/2}z^{u-1/2}f(z) = r(z)\ . \tag{9}$$

The homogeneous solutions of the latter are also confluent hypergeometric functions. They are known as Whittaker functions. The cases $r(z) = z^{u-1/2}$ and $z^{\nu-1/2}$ are treated in Chapter 5. This chapter also contains material on the particular integrals of the nonhomogeneous equation for the functions of the paraboloid of revolution and for the parabolic cylinder functions.

Chapters 6 and 7 take up (5) with $p = 2$, $q = 1$. Thus the homogeneous solutions are Gaussian hypergeometric functions which include Legendre functions. Here several forms of $f(x)$, as given by (5) with $\lambda = 0$, are considered.

Generalizations of some of the material in the previous chapters is given in Chapter 8 by putting $f(x) = t_{p-1}F_{q-1}(\alpha_1 + 1, \cdots, \alpha_{p-1} + 1; \rho_1, \cdots, \rho_{q-1}; \frac{1}{2}x)$, t a constant. Thus if $p = q = 1$, $f(x) = te^{x/2}$, see Chapter 4; and if $p = 2$ and $q = 1$, $f(t) = t(1 - x/2)^{-\alpha_1-1}$, see Chapter 6.

The homogeneous differential equations encountered in Chapters 3–8 have at most three regular singularities. Now second-order differential equations with four regular singularities at $z = 0, 1, a$ and ∞ can be reduced to a canonical form which goes by the name of Heun. This includes the differential equation for Lamé and Mathieu functions. Chapter 9 discusses Heun's equation, call it $H(D)y = 0$, and particular integrals of $H(D)y = [(z - 1)(z - a)]^{-1}z^{\sigma-2}$.

Nonhomogeneous linear partial differential equations is the subject of Chapter 10. Chapter 11 is devoted to nonhomogeneous partial differential equations of mathematical physics. In particular, the methods of Chapter 10 are used to solve the equations:

(10) $\Delta V = f(x, y, z)$,

(11) $\Delta V - c^{-2}\, \partial^2 V/\partial t^2 = f(x, y, z, t)$,

(12) $\Delta V - k^{-1}\, \partial V/\partial t = f(x, y, z, t)$,

where Δ is the Laplacian, that is

(13) $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$.

The Laplacian is expressed in terms of some commonly used systems of orthogonal curvilinear coordinates, and if V is separable, then for each coordinate system the three resulting differential equations are presented. If $f(x, y, z)$ is separable and has a certain form in a given coordinate system, then a solution of (10) is also separable and the three resulting ordinary nonhomogeneous differential equations are set forth. Special cases relating this material to that of the earlier chapters are noted. Similar material for (11), (12) is also developed.

Y. L. L.

1. A. Erdélyi, W. Magnus, F. Oberhettinger & F. G. Tricomi, *Higher Transcendental Functions*, Vols. 1, 2, McGraw-Hill, New York, 1953.
2. G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge Univ. Press, Cambridge, 1945.
3. Y. L. Luke, *Integrals of Bessel Functions*, McGraw-Hill, New York, 1962.

12 [7, 9].—M. Lal, *First* 39000 *Decimal Digits of* $\sqrt{2}$, deposited in UMT file.

Previous results on the decimal expansion of $\sqrt{2}$ to 19600D [1] are here extended to 39000D.

In the present investigation, the Newton-Raphson method was used to find an improved approximation to $\sqrt{2}$. Let x_1 be an approximate value of $\sqrt{2}$, then a value of x_2, accurate to twice the number of digits as x_1, is given by

$$x_2 = \tfrac{1}{2}\, x_1 + x_1^{-1} .$$

Here x_1 was known to 19600D, and in order to double the accuracy the reciprocal of x_1 must be carried to 2×19600 digits. The process of dividing 1 by x_1 was carried out on the 1620, Model II, 60K at Queen's University in two parts, and 39075D of x_2 were obtained. The accuracy of the value of x_2 was checked by squaring the output of 39076 digits with the decimal point disregarded. This multiplication, which was carried out in five sections of approximately 8K digits each, showed a 1 followed by an unbroken string of 39074 nines. This test establishes that this value of $\sqrt{2}$ is accurate to 39074D. The first 39000D are recorded here.

In order to examine the internal randomness of digits in an unsophisticated way, the frequency distribution of digits in 39 blocks of 1000 digits, and also in the total 39000 digits, was computed. The chi-square test for the goodness of fit reveals no abnormal behavior in the distribution of digits in this sample. These data are also recorded here.

Author's Summary

1. M. Lal, *Expansion of* $\sqrt{2}$ *to* 19600 *Decimals*, reviewed in *Math. Comp.*, v. 21, 1967, pp. 258–259, RMT 17.